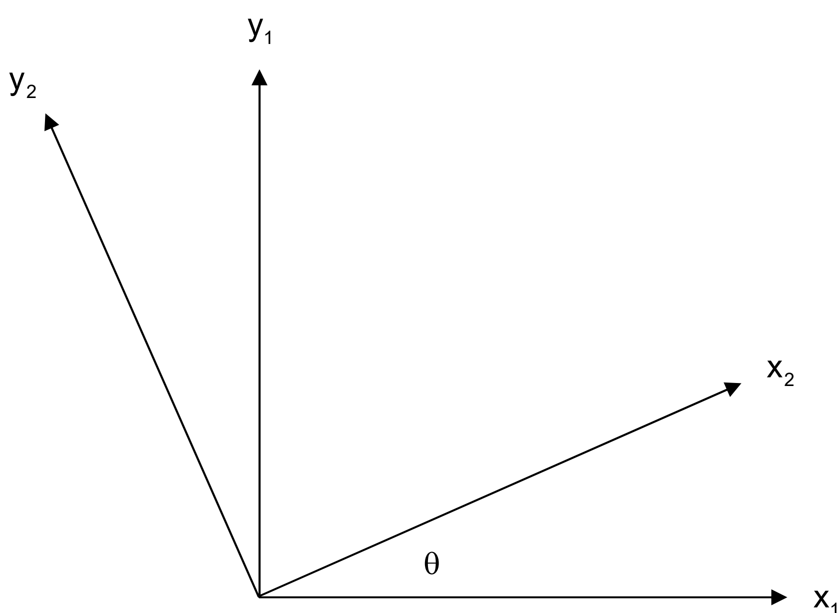
Transformation of a stress tensor by a plane rotation[1]

Define a stress tensor S_i where the subscript i will indicate the coordinate system that the stresses are in. Then:

$$S_i = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}_i \tag{1.1}$$

where $\sigma_{ij} = \sigma_{ji}$ and, for example, σ_{xy} is the x-y plane shear stress (also called τ_{xy})

We would like to express the stresses in coordinate system 2 as a transformation from system 1 where system 2 is rotated through an angle θ in the x-y plane from system 1 as shown below (note: z is perpendicular to the diagram):



Since stress is a tensor, the coordinate transformation follows the rule:

$$S_2 = T_{12}^T S_1 T_{12} \tag{1.2}$$

where T_{12} is the plane coordinate transformation matrix which expresses coordinate system 1 in terms of coordinate system 2:

[1] Dr Bill Case, Oct 4 2007

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \tag{1.3}$$

or

$$X_1 = T_{12} X_2 \tag{1.4}$$

where:

$$X_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \quad \text{and} \quad T_{12} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.5}$$

Performing the triple matrix product in equation 1.2 we can write a vector of the 6 stresses in coordinate system 2 in terms of those in system 1. Changing notation to the more familiar terms for stresses this becomes:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_2 = \begin{bmatrix} c^2 & s^2 & 0 & 2sc & 0 & 0 \\ s^2 & c^2 & 0 & -2sc & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -sc & sc & 0 & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_1 \tag{1.6}$$

where

$$\begin{aligned} s &= \sin\theta \\ c &= \cos\theta \\ sc &= \sin\theta\cos\theta \end{aligned} \tag{1.7}$$

To demonstrate that equation 1.6 gives the more familiar standard equations for 2D stress transformations let us rewrite the 1st, 2nd and 3rd rows from equation 1.6 using some familiar trigonometric relationships:

$$\begin{aligned} \sigma_{x_2} &= \sigma_{x_1}\cos^2\theta + \sigma_{y_1}\sin^2\theta + 2\tau_{xy_1}\sin\theta\cos\theta \\ &= \frac{\sigma_{x_1}}{2}(1+\cos 2\theta) + \frac{\sigma_{y_1}}{2}(1-\cos 2\theta) + \tau_{xy_1}\sin 2\theta \\ &= \frac{(\sigma_{x_1}+\sigma_{y_1})}{2} + \frac{(\sigma_{x_1}-\sigma_{y_1})}{2}\cos 2\theta + \tau_{xy_1}\sin 2\theta \end{aligned} \tag{1.8}$$

$$\begin{aligned}\sigma_{y_2} &= \sigma_{x_1}\sin^2\theta + \sigma_{y_1}\cos^2\theta - 2\tau_{xy_1}\sin\theta\cos\theta \\ &= \frac{\sigma_{x_1}}{2}(1-\cos 2\theta) + \frac{\sigma_{y_1}}{2}(1+\cos 2\theta) - \tau_{xy_1}\sin 2\theta \\ &= \frac{(\sigma_{x_1}+\sigma_{y_1})}{2} - \frac{(\sigma_{x_1}-\sigma_{y_1})}{2}\cos 2\theta - \tau_{xy_1}\sin 2\theta\end{aligned} \tag{1.9}$$

$$\begin{aligned}\tau_{xy_2} &= -\sigma_{x_1}\sin\theta\cos\theta + \sigma_{y_1}\sin\theta\cos\theta + \tau_{xy_1}(\cos^2\theta - \sin^2\theta) \\ &= -\frac{\sigma_{x_1}-\sigma_{y_1}}{2}\sin 2\theta + \tau_{xy_1}\cos 2\theta\end{aligned} \tag{1.10}$$

Equations 1.8, 1.9 and 1.10 are expressed in the more familiar plane stress coordinate transformations so that equation 1.6 is validated for at least this simple test.

Note that the transverse shears, τ_{yz} and τ_{yz}, can be written in a more convenient format. From the 5th and 6th rows of equation 1.6:

$$\begin{Bmatrix}\tau_{yz} \\ \tau_{zx}\end{Bmatrix}_2 = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\tau_{yz} \\ \tau_{zx}\end{Bmatrix}_1 \tag{1.11}$$

Rearranging the rows and renaming τ_{zx} as τ_{xz}, equation 1.11 can be written as:

$$\begin{Bmatrix}\tau_{xz} \\ \tau_{yx}\end{Bmatrix}_2 = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\tau_{xz} \\ \tau_{yx}\end{Bmatrix}_1 \tag{1.12}$$

τ_{xz} and τ_{yx} are the two shear stresses that are quoted in the literature as plate transverse shear stresses.

The above equations apply equally well to strain coordinate transformations