

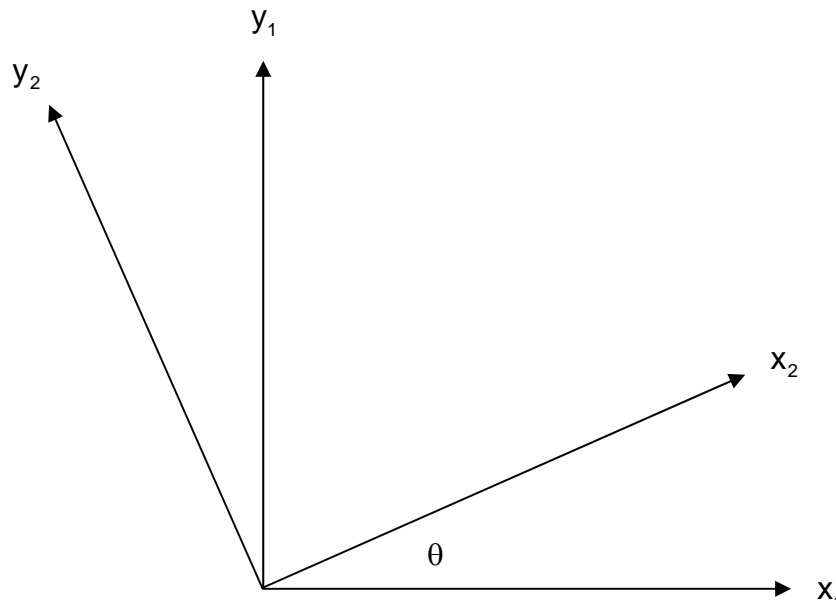
## Transformation of a stress tensor by a plane rotation<sup>1</sup>

Define a stress tensor  $S_i$  where the subscript  $i$  will indicate the coordinate system that the stresses are in. Then:

$$S_i = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}_i \quad (1.1)$$

where  $\sigma_{ij} = \sigma_{ji}$  and, for example,  $\sigma_{xy}$  is the x-y plane shear stress (also called  $\tau_{xy}$ )

We would like to express the stresses in coordinate system 2 as a transformation from system 1 where system 2 is rotated through an angle  $\theta$  in the x-y plane from system 1 as shown below (note: z is perpendicular to the diagram):



Since stress is a tensor, the coordinate transformation follows the rule:

$$S_2 = T_{12}^T S_1 T_{12} \quad (1.2)$$

where  $T_{12}$  is the plane coordinate transformation matrix which expresses coordinate system 1 in terms of coordinate system 2:

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<sup>1</sup> Dr Bill Case, Oct 4 2007

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \quad (1.3)$$

or

$$X_1 = T_{12} X_2 \quad (1.4)$$

where:

$$X_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \quad \text{and} \quad T_{12} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.5)$$

Performing the triple matrix product in equation 1.2 we can write a vector of the 6 stresses in coordinate system 2 in terms of those in system 1. Changing notation to the more familiar terms for stresses this becomes:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_2 = \begin{bmatrix} c^2 & s^2 & 0 & 2sc & 0 & 0 \\ s^2 & c^2 & 0 & -2sc & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -sc & sc & 0 & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_1 \quad (1.6)$$

where

$$\begin{aligned} s &= \sin \theta \\ c &= \cos \theta \\ sc &= \sin \theta \cos \theta \end{aligned} \quad (1.7)$$

To demonstrate that equation 1.6 gives the more familiar standard equations for 2D stress transformations let us rewrite the 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> rows from equation 1.6 using some familiar trigonometric relationships:

$$\begin{aligned} \sigma_{x_2} &= \sigma_{x_1} \cos^2 \theta + \sigma_{y_1} \sin^2 \theta + 2\tau_{xy_1} \sin \theta \cos \theta \\ &= \frac{\sigma_{x_1}}{2} (1 + \cos 2\theta) + \frac{\sigma_{y_1}}{2} (1 - \cos 2\theta) + \tau_{xy_1} \sin 2\theta \\ &= \frac{(\sigma_{x_1} + \sigma_{y_1})}{2} + \frac{(\sigma_{x_1} - \sigma_{y_1})}{2} \cos 2\theta + \tau_{xy_1} \sin 2\theta \end{aligned} \quad (1.8)$$

$$\begin{aligned}
\sigma_{y_2} &= \sigma_{x_1} \sin^2 \theta + \sigma_{y_1} \cos^2 \theta - 2\tau_{xy_1} \sin \theta \cos \theta \\
&= \frac{\sigma_{x_1}}{2} (1 - \cos 2\theta) + \frac{\sigma_{y_1}}{2} (1 + \cos 2\theta) - \tau_{xy_1} \sin 2\theta \\
&= \frac{(\sigma_{x_1} + \sigma_{y_1})}{2} - \frac{(\sigma_{x_1} - \sigma_{y_1})}{2} \cos 2\theta - \tau_{xy_1} \sin 2\theta
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
\tau_{xy_2} &= -\sigma_{x_1} \sin \theta \cos \theta + \sigma_{y_1} \sin \theta \cos \theta + \tau_{xy_1} (\cos^2 \theta - \sin^2 \theta) \\
&= -\frac{\sigma_{x_1} - \sigma_{y_1}}{2} \sin 2\theta + \tau_{xy_1} \cos 2\theta
\end{aligned} \tag{1.10}$$

Equations 1.8, 1.9 and 1.10 are expressed in the more familiar plane stress coordinate transformations so that equation 1.6 is validated for at least this simple test.

Note that the transverse shears,  $\tau_{yz}$  and  $\tau_{yx}$ , can be written in a more convenient format. From the 5<sup>th</sup> and 6<sup>th</sup> rows of equation 1.6:

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_1 \tag{1.11}$$

Rearranging the rows and renaming  $\tau_{zx}$  as  $\tau_{xz}$ , equation 1.11 can be written as:

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yx} \end{Bmatrix}_2 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \tau_{xz} \\ \tau_{yx} \end{Bmatrix}_1 \tag{1.12}$$

$\tau_{xz}$  and  $\tau_{yx}$  are the two shear stresses that are quoted in the literature as plate transverse shear stresses.

The above equations apply equally well to strain coordinate transformations